Kinetically driven glassy transition in an exactly solvable toy model with reversible mode coupling mechanism and trivial statics

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Abstract. We propose a toy model with reversible mode coupling mechanism and with trivial Hamiltonian (and hence trivial statics). The model can be analyzed exactly without relying upon uncontrolled approximation such as the factorization approximation employed in the current MCT. We show that the model exhibits a kinetically driven transition from an ergodic phase to nonergodic phase. The nonergodic state is the nonequilibrium stationary solution of the Fokker-Planck equation for the distribution function of the model.

1. Introduction

First-principles understanding on the rich dynamic phenomena and the nature of the liquid-glass transition still remains as a challenging aim [1]. As the only existing first-principle theory, the mode coupling theory (MCT) of supercooled liquids and the glass transition enjoyed considerable success in describing the dynamics of weakly supercooled regime of liquids [2]. Notwithstanding this surprising success, there are the following several unresolved issues concerning the basis of MCT: (a) A crucial ingredient of MCT is the factorization approximation which replaces the four-body time correlation functions by the product of two-body time correlation functions. This approximation is completely uncontrolled and its region of validity is *a priori* unknown. (b) The idealized MCT predicts a sharp dynamic transition to a nonergodic state at a certain temperature. But MCT does not provide any information on the nature of this nonergodic state. (c) The physical picture of the so called hopping processes in an extended version of MCT is still lacking.

In recent years, possible deep connection between the structural glass and a class of spin glass models has been pointed out [3]. In particular, the Langevin dynamics of the spherical p-spin model can be analyzed exactly in the thermodynamic limit due to the mean field nature of the model (i.e., full connectivity of the spins) [4, 5]. This analysis shows that the dynamic equation for the spin correlation function in equilibrium for $p = 3$ has the same form as in the Leutheusser's schematic mode coupling equation for the density correlator [6]. The sharp dynamic transition observed in this class of models are driven by the dissipative nonlinearity in the equation of motion which originates from the nonlinear Hamiltonian [7]. In contrast to this, the

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glassy behavior in the above-mentioned MCT (as well as in our model given below) is driven by the *reversible* nonlinearity [8] which is dynamically generated and hence a non-trivial Hamiltonian is not necessary.

With these situations, we thought that it is important to develop a toy model with following three ingredients:

- reversible mode coupling mechanism
- trivial statics
- mean-field type so that the model can be exactly solvable.

We have proposed such a toy model in a recent publication [9]. Here we further analyze the model. The model yields the self-consistent equations for the relevant correlation functions of the type familiar in the mode coupling theories of supercooled liquid and glass transition, where the strength of the hopping processes can be readily tuned. In the sense that the glassy behavior in this toy model is induced by the kinetics of the reversible mode coupling mechanism, our model is similar in spirit to the kinetically constrained models, the theme of the present workshop.

2. Model

Our model is defined as the following Langevin equations for the N-component density variable $a_i(t)$ with $i = 1, 2, \cdots, N$ and the M-component velocity variable b_α with $\alpha = i, 2, \cdots, M$. Here and after we will use Roman indices for the component of a and Greek for that of b.

$$\dot{a}_i = K_{i\alpha} b_\alpha + \frac{\omega}{\sqrt{N}} J_{ij\alpha} a_j b_\alpha \tag{1}$$

$$\dot{b}_\alpha = -\gamma b_\alpha - \omega^2 K_{j\alpha} a_j - \frac{\omega}{\sqrt{N}} J_{ij\alpha}(\omega^2 a_i a_j - T\delta_{ij}) + f_\alpha \tag{2}$$

$$< f_\alpha(t) >= 0, \qquad < f_\alpha(t) f_\beta(t') >= 2\gamma T \delta_{\alpha\beta} \delta(t - t') \tag{3}$$

where the summation is implied for repeated indices. Here γ is the decay rate of the velocity b_α and ω gives the j-independent frequency of oscillation of the density a_j. The thermal noise $f_\alpha(t)$ are independent gaussian random variables with zero mean and variance $2\gamma T$, T being the temperature of the heat bath with which the system has a thermal contact. The choice of this variance guarantees the proper equilibration of the system. The $N \times M$ matrix $K_{i\alpha}$ plays an important role in the model and for later purpose we impose the (one-sided) orthogonality

$$K_{i\alpha} K_{i\beta} = \delta_{\alpha\beta}, \qquad K_{i\alpha} K_{j\alpha} \neq \delta_{ij} \tag{4}$$

where the last equation is due to the inequality $M < N$. For $M = N$ we can impose an additional condition $K_{i\alpha} = \delta_{i\alpha}$ and hence trivially $K_{i\alpha} K_{j\alpha} = \delta_{ij}$. We also note that $K_{i\alpha}$ governs linearized reversible dynamics of the model with the dynamical matrix $\mathbf{\Omega}$ given by $\Omega_{ij} \equiv \omega^2 K_{i\alpha} K_{j\alpha}$. The reversible nonlinear mode coupling terms are the ones involving the mode coupling coefficients $J_{ij\alpha}$ which are chosen to be quenched (time-independent) gaussian random variables with the following properties:

$$\begin{aligned} \overline{J_{ij\alpha}}^J &= 0, \\ \overline{J_{ij\alpha} J_{kl\beta}}^J &= \frac{g^2}{N} \Big[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{\alpha\beta} + K_{i\beta}(K_{k\alpha}\delta_{jl} + K_{l\alpha}\delta_{jk}) \\ &\quad + K_{j\beta}(K_{k\alpha}\delta_{il} + K_{l\alpha}\delta_{ik}) \Big] \end{aligned} \tag{5}$$

where $\overline{\cdots}^J$ denotes average over the J's. Note that there is no thermal noise which acts directly on the density variable in (1). This is because the model is constructed so as to mimic the dynamics of fluid. Equation (1) is analogous to the equation of continuity of fluid and Eq.(2) is like the equation of motion where the right hand side is like the force acting on a fluid element. In constructing this model, we were motivated by the works [10, 11] in which random coupling models involving an infinite component order parameter have been shown to be exactly analyzed by mean-field-type concepts. We will thus eventually take N and M infinite with the ratio $\delta^* \equiv M/N$ kept finite.

One can derive from the Langevin equations (1)-(3) the corresponding Fokker-Planck equation for the probability distribution function $D(\{a\},\{b\},t)$ for our variable set denoted as $\{a\},\{b\}$ as follows

$$\partial_t D(\{a\},\{b\},t) = \hat{L} D(\{a\},\{b\},t) \tag{6}$$

where the Fokker-Planck operator is given by $\hat{L} = \hat{L}_0 + \hat{L}_1 + \hat{L}_{MC}$ with

$$\begin{aligned}
&\hat{L}_0 \equiv \frac{\partial}{\partial b_\alpha}\gamma\left(T\frac{\partial}{\partial b_\alpha} + b_\alpha\right), \quad \hat{L}_1 \equiv K_{j\alpha}\left(-\frac{\partial}{\partial a_j}b_\alpha + \frac{\partial}{\partial b_\alpha}\omega^2 a_j\right), \\
&\hat{L}_{MC} \equiv \frac{1}{\sqrt{N}} J_{ij\alpha}\left(-\frac{\partial}{\partial a_i}\omega a_j b_\alpha + \frac{\partial}{\partial b_\alpha}\omega(\omega^2 a_i a_j - T\delta_{ij})\right)
\end{aligned} \tag{7}$$

It is then easy to show that the *equilibrium* stationary distribution (i.e., $\hat{L}D_e(a,b) = 0$) is given by

$$D_e(\{a\},\{b\}) = cst.\mathrm{e}^{-\sum_{j=1}^{N}\frac{\omega^2}{2T}a_j^2 - \sum_{\alpha=1}^{M}\frac{1}{2T}b_\alpha^2} \tag{8}$$

where $cst.$ is the normalization factor.

3. Analysis and discussion

For the subsequent analysis it is most convenient to introduce the following generating functional

$$\hat{Z}\{h^a,\hat{h}^a,h^b,\hat{h}^b\} \equiv \int d\{a\}\int d\{b\}\int d\{\hat{a}\}\int d\{\hat{b}\}\mathrm{e}^{\mathrm{i}\int dt(h_j^a a_j + \hat{h}_j^a \hat{a}_j + h_\alpha^b b_\alpha + \hat{h}_\alpha^b \hat{b}_\alpha)}\mathrm{e}^{\hat{\mathcal{S}}} \tag{9}$$

where the integrals are the functional integrals over the variable sets $\{a\},\{\hat{a}\},\{b\},\{\hat{b}\}$ and the h's and the $\hat{h}$'s the conjugate source fields. The action $\hat{\mathcal{S}}$ was decomposed into two parts $\hat{\mathcal{S}}_0$ and $\hat{\mathcal{S}}_I$ which take the form

$$\hat{\mathcal{S}}_0 = \int dt\left\{\mathrm{i}\hat{a}_i\big(\dot{a}_i - K_{i\alpha}b_\alpha\big) + \mathrm{i}\hat{b}_\alpha\big(\dot{b}_\alpha + \gamma b_\alpha + \omega^2 K_{i\alpha}a_i - f_\alpha\big)\right\}(t) \tag{10}$$

$$\hat{\mathcal{S}}_I = J_{jk\alpha}\hat{X}_{jk\alpha} \tag{11}$$

$$\hat{X}_{jk\alpha} \equiv \frac{\omega}{\sqrt{N}}\int dt\left\{-\mathrm{i}\hat{a}_j a_k b_\alpha + \mathrm{i}\hat{b}_\alpha\omega^2 a_j a_k\right\}(t) \tag{12}$$

where we have dropped the term $T\delta_{ij}$ coming from eq.(2) since this term is negligible in the limit of infinite M and N. The functional determinant associated with the Langevin equations (1)-(3) which should appear in the integrand of the generating functional $\hat{Z}$ was equated to unity assuming the Itô calculus [12]. The various correlation functions and response functions are obtained by taking various functional derivatives of $\ln Z\{h^a,\hat{h}^a,h^b,\hat{h}^b\}$ with respect to h's and $\hat{h}$'s and setting them equal to

zero in the end in the standard way, where Z is the generating functional $\hat{Z}$, averaged over the f's and the J's.

We now note that the replacements $\mathrm{i}\hat{a}_j \to (\omega^2/T)a_j$, $\quad \hat{b}_\alpha \to b_\alpha/T$ in $\hat{X}_{jk\alpha}$ leads to $\hat{X}_{jk\alpha} = 0$. Hence we can rewrite $\hat{X}_{jk\alpha}$ also as

$$\hat{X}_{jk\alpha} = \tilde{X}_{jk\alpha} \equiv \frac{\omega}{\sqrt{N}} \int dt \left\{ -\mathrm{i}\tilde{a}_j a_k b_\alpha + \mathrm{i}\omega^2 \tilde{b}_\alpha a_j a_k \right\}(t) \tag{13}$$

where $\mathrm{i}\tilde{a}_i \equiv \mathrm{i}\hat{a}_i + (\omega^2/T)a_i$ and $\mathrm{i}b_\alpha \equiv \mathrm{i}\hat{b}_\alpha + b_\alpha/T$.

We now obtain for this toy model the equilibrium correlation functions defined as

$$\begin{aligned}
&C_a(t-t') \equiv \frac{1}{N} < a_j(t) a_j(t') >, \quad C_{ab}(t-t') \equiv \frac{1}{M} K_{j\alpha} < a_j(t) b_\alpha(t') >, \\
&C_{ba}(t-t') \equiv \frac{1}{M} K_{j\alpha} < b_\alpha(t) a_j(t') >, \quad C_b(t-t') \equiv \frac{1}{M} < b_\alpha(t) b_\alpha(t') >, \\
&\qquad C_a^K(t-t') \equiv \frac{1}{M} K_{i\alpha} K_{j\alpha} < a_i(t) a_j(t') >
\end{aligned} \tag{14}$$

It turns out that we need to have the last correlation function to close the self-consistent set of equations for the correlators when $M < N$. Note that for the case $M = N$, if $K_{i\alpha} = \delta_{i\alpha}$ is imposed, then $C_a^K(t-t') = C_a(t-t')$. The corresponding response functions can be defined as

$$\begin{aligned}
&G_a(t-t') \equiv \frac{1}{N} < a_j(t) \mathrm{i}\hat{a}_j(t') >, \quad G_{ab}(t-t') \equiv \frac{1}{M} K_{j\alpha} < a_j(t) \mathrm{i}\hat{b}_\alpha(t') >, \\
&G_{ba}(t-t') \equiv \frac{1}{M} K_{j\alpha} < b_\alpha(t) \mathrm{i}\hat{a}_j(t') >, \quad G_b(t-t') \equiv \frac{1}{M} < b_\alpha(t) \mathrm{i}\hat{b}_\alpha(t') >, \\
&\qquad G_a^K(t-t') \equiv \frac{1}{M} K_{i\alpha} K_{j\alpha} < a_i(t) \mathrm{i}\hat{a}_j(t') >
\end{aligned} \tag{15}$$

Since we have a Gaussian stationary solution, we get the fluctuation-dissipation relationships (FDR) of the form [13]

$$\begin{aligned}
&G_a(t-t') = -\theta(t-t') \frac{\omega^2}{T} C_a(t-t'), \quad G_{ab}(t-t') = -\theta(t-t') \frac{1}{T} C_{ab}(t-t'), \\
&G_{ba}(t-t') = -\theta(t-t') \frac{\omega^2}{T} C_{ba}(t-t'), \quad G_b(t-t') = -\theta(t-t') \frac{1}{T} C_b(t-t'), \\
&\qquad G_a^K(t-t') = -\theta(t-t') \frac{\omega^2}{T} C_a^K(t-t')
\end{aligned} \tag{16}$$

where $\theta(t)$ is the unit step function: $\theta(t) = 1$ for $t \geq 0$ and 0 otherwise. Note that this form of the FDR is rather unusual since the FDR usually takes the form $G(t) = -\theta(t) \partial_t C(t)/T$.

Another useful property arising from the causality and the above FDR is the following property

$$< \hat{A}(t) X(t') > = < X(t) \tilde{A}(t') > = 0 \quad \text{for} \quad t \geq t' \tag{17}$$

for $A(t) = (a(t), b(t))$ and an arbitrary function $X(t) = X(a(t), b(t), \hat{a}(t), \hat{b}(t))$.

We now take averages of $\hat{Z}$ over the thermal noise f_α and the quenched random coupling $J_{ij\alpha}$. In so doing we use the following properties which hold for the gaussian random variables:

$$\begin{aligned}
&\left\langle \mathrm{e}^{-\mathrm{i} \int dt \hat{b}_\alpha(t) f_\alpha(t)} \right\rangle = \mathrm{e}^{-\gamma T \int dt \hat{b}_\alpha(t)^2} \\
&\overline{\mathrm{e}^{J_{jk\alpha} \hat{X}_{jk\alpha}}}^J = \mathrm{e}^{\frac{1}{2} \overline{J_{jk\alpha} J_{lm\beta}}^J \hat{X}_{jk\alpha} \hat{X}_{lm\beta}}
\end{aligned} \tag{18}$$

Defining the actions S_0 and S_I as

$$\mathrm{e}^{\mathcal{S}_0} \equiv \left\langle \mathrm{e}^{\hat{S}_0} \right\rangle, \qquad \mathrm{e}^{\mathcal{S}_I} \equiv \overline{\mathrm{e}^{\hat{S}_I}}^J, \tag{19}$$

we obtain

$$\begin{aligned}\mathcal{S}_0 &= \int dt \left\{ \mathrm{i}\hat{a}_i(\dot{a}_i - K_{i\alpha}b_\alpha)(t) + \mathrm{i}\hat{b}_\alpha(\dot{b}_\alpha + \gamma b_\alpha + \omega^2 K_{i\alpha}a_i)(t) - \gamma T \hat{b}_\alpha^2(t) \right\} \\ &= \int dt \left\{ \mathrm{i}\hat{a}_i \left(\frac{T}{\omega^2} \mathrm{i}\dot{\tilde{a}}_i - TK_{i\alpha}\mathrm{i}\tilde{b}_\alpha \right)(t) + \mathrm{i}\hat{b}_\alpha \left(T\mathrm{i}\dot{\tilde{b}}_\alpha + TK^T_{\alpha i}\mathrm{i}\tilde{a}_i + \gamma T \mathrm{i}\hat{b}_\alpha \right)(t) \right\}\end{aligned} \tag{20}$$

where the last line is obtained using the property (17). Now we have to deal with the interaction part $S_I = \overline{J_{jk\alpha}J_{lm\beta}}^J \hat{X}_{jk\alpha}\hat{X}_{lm\beta}/2$. One can show that in the limit of $M, N \to \infty$ fluctuations can be neglected so that quantities like $a_j(t)a_j(t')/N$ etc. are replaced by $C_a(t,t')$, etc. The interaction part $\mathcal{S}_I$ then becomes gaussianized in the limit of $M, N \to \infty$. The final expression for $\mathcal{S}_I$ is then given by

$$\begin{aligned}\mathcal{S}_I = \int dt \Big\{ & i\hat{a}_i(t)\frac{T}{\omega^2}\Sigma_{aa} \otimes i\tilde{a}_i(t) + K_{i\alpha} i\hat{a}_i(t) T\Sigma_{ab} \otimes i\tilde{b}_\alpha(t) \\ &+ K_{i\alpha} i\hat{b}_\alpha(t) \frac{T}{\omega^2}\Sigma_{ba} \otimes i\tilde{a}_i(t) + i\hat{b}_\alpha(t) T\Sigma_{bb} \otimes i\tilde{b}_\alpha(t) \Big\}\end{aligned} \tag{21}$$

where $\Sigma \otimes a(t) \equiv \int_{-\infty}^t dt'\, \Sigma(t-t')a(t')$ etc. Here the kernels Σ's are given by

$$\begin{aligned}\Sigma_{aa}(t-t') &\equiv \delta^* \frac{g^2\omega^4}{T}\big(C_a(t-t')C_b(t-t') + \delta^* C_{ab}(t-t')C_{ba}(t-t')\big), \\ \Sigma_{ab}(t-t') &\equiv -2\delta^* \frac{g^2\omega^4}{T} C_a(t-t')C_{ba}(t-t') \\ \Sigma_{ba}(t-t') &\equiv -2\delta^* \frac{g^2\omega^6}{T} C_a(t-t')C_{ab}(t-t'), \\ \Sigma_{bb}(t-t') &\equiv \frac{2g^2\omega^6}{T} C_a(t-t')^2\end{aligned} \tag{22}$$

Three kernels Σ_{aa}, Σ_{ab}, and Σ_{ba} comes from the nonlinear coupling term in the original Langevin equation (1), and the kernel Σ_{bb} arises from the density nonlinearity in (2). We also note that the correlator $C_a^K(t,t')$ is not involved in the Σ's.

From the effective gaussian action $\mathcal{S}_{eff} \equiv \mathcal{S}_0 + \mathcal{S}_I$ we can readily write down the following linearized Langevin equations for a_i and b_α

$$\dot{a}_i(t) = K_{i\alpha}b_\alpha(t) - \Sigma_{aa} \otimes a_i(t) - K_{i\alpha}\Sigma_{ab} \otimes b_\alpha(t) + f_i^a(t) \tag{23}$$

$$\dot{b}_\alpha(t) = -\gamma b_\alpha(t) - \omega^2 K_{i\alpha}a_i(t) - K_{i\alpha}\Sigma_{ba} \otimes a_i(t) - \Sigma_{bb} \otimes b_\alpha(t) + f_\alpha^b(t) \tag{24}$$

where f^a and f^b are the effective thermal noises whose correlations are given by

$$\begin{aligned} < f_i^a(t) f_j^a(t') > &= \frac{T}{\omega^2}[\Sigma_{aa}(tt') + \Sigma_{aa}(t't)]\delta_{ij} \\ < f_i^a(t) f_\alpha^b(t') > &= K_{i\alpha}T[\Sigma_{ab}(tt') + \frac{1}{\omega^2}\Sigma_{ba}(t't)] \\ < f_\alpha^b(t) f_i^a(t') > &= K_{i\alpha}T[\Sigma_{ba}(tt') + \frac{1}{\omega^2}\Sigma_{ab}(tt')] \\ < f_\alpha^b(t) f_\beta^b(t') > &= \Big(2\gamma T\delta(t-t') + T[\Sigma_{bb}(tt') + \Sigma_{bb}(t't)] \Big)\delta_{\alpha\beta}\end{aligned} \tag{25}$$

Now we are ready to obtain a set of self-consistent equations for the *five* correlators from the linearized Langevin equations. By multiplying (23) by $a_i(0)/N$ and (24) by $K_{i\alpha} a_i(0)/M$ and averaging over the effective thermal noise, we obtain

$$\dot{C}_a(t) = \delta^* C_{ba}(t) - \Sigma_{aa} \otimes C_a(t) - \delta^* \Sigma_{ab} \otimes C_{ba}(t) \tag{26}$$

$$\dot{C}_{ba}(t) = -\gamma C_{ba}(t) - \omega^2 C_a^K(t) - \Sigma_{ba} \otimes C_a^K(t) - \Sigma_{bb} \otimes C_{ba}(t) \tag{27}$$

where we used the causality requirements $< f_i^a(t) a_i(0) >= 0$ and $K_{i\alpha} < f_\alpha^b(t) a_i(0) >= 0$. Note that the correlator $C_a^K(t)$ appears in the equation for $C_{ba}(t)$. In order to obtain the equation for $C_a^K(t)$, we multiply (23) by $K_{i\beta} K_{j\beta} a_j(0)/N$ and take thermal average. Then we obtain

$$\dot{C}_a^K(t) = C_{ba}(t) - \Sigma_{aa} \otimes C_a^K(t) - \Sigma_{ab} \otimes C_{ba}(t) \tag{28}$$

Similarly by multiplying (23) and (24) by $K_{i\beta} b_\beta(0)/M$ and $b_\alpha(0)/M$, respectively, and performing the thermal average we obtain the following equations for $C_{ab}(t)$ and $C_b(t)$

$$\dot{C}_{ab}(t) = C_b(t) - \Sigma_{aa} \otimes C_{ab}(t) - \Sigma_{ab} \otimes C_b(t), \tag{29}$$

$$\dot{C}_b(t) = -\gamma C_b(t) - \omega^2 C_{ab}(t) - \Sigma_{ba} \otimes C_{ab}(t) - \Sigma_{bb} \otimes C_b(t) \tag{30}$$

The equations (26)-(30) constitute the self-consistent equations for the 5 correlators $C_a(t)$, $C_{ba}(t)$, $C_a^K(t)$, $C_{ab}(t)$, and $C_b(t)$. This set of equations can be solved numerically with the initial conditions $C_a(0) = C_a^K(0) = T/\omega^2$, $C_{ab}(0) = C_{ba}(0) = 0$, and $C_b(0) = T$.

In analytic side, it is very convenient to work with the equations of the Laplace transformed correlation functions defined as $C^L(z) \equiv \int_0^\infty dt\, \mathrm{e}^{-zt}\, C(t)$. Performing the Laplace transformation of the self-consistent equations we obtain

$$zC_a^L(z) = \frac{T}{\omega^2} + (1 - \Sigma_{ab}^L(z))\delta^* C_{ba}^L(z) - \Sigma_{aa}^L(z) C_a^L(z) \tag{31}$$

$$zC_{ba}^L(z) = -(\gamma + \Sigma_{bb}^L(z)) C_{ba}^L(z) - (\omega^2 + \Sigma_{ba}^L(z)) C_a^{KL}(z) \tag{32}$$

$$zC_a^{KL}(z) = \frac{T}{\omega^2} + (1 - \Sigma_{ab}^L(z)) C_{ba}^L(z) - \Sigma_{aa}^L(z) C_a^{KL}(z) \tag{33}$$

$$zC_{ab}^L(z) = (1 - \Sigma_{ab}^L(z)) C_b^L(z) - \Sigma_{aa}^L(z) C_{ab}^L(z) \tag{34}$$

$$zC_b^L(z) = T - (\omega^2 + \Sigma_{ba}^L(z)) C_{ab}^L(z) - (\gamma + \Sigma_{bb}^L(z)) C_b^L(z) \tag{35}$$

From (31)-(33), we obtain $C_a^L(z)$, $C_a^{KL}(z)$, and $C_{ba}^L(z)$ in terms of Σ's as follows:

$$C_a^L(z) = \frac{T}{\omega^2} \frac{1}{z + \Sigma_{aa}^L(z)} \left[1 - \delta^* \frac{\omega^2 (1 - \Sigma_{ab}^L(z))^2}{(z + \Sigma_{aa}^L(z))(z + \gamma + \Sigma_{bb}(z)) + \omega^2 (1 - \Sigma_{ab}^L(z))^2} \right] \tag{36}$$

$$C_a^{KL}(z) = \frac{T}{\omega^2} \left[z + \Sigma_{aa}^L(z) + \frac{\omega^2 (1 - \Sigma_{ab}^L(z))^2}{z + \gamma + \Sigma_{bb}(z)} \right]^{-1} \tag{37}$$

$$C_{ba}^L(z) = -\frac{T(1 - \Sigma_{ab}^L(z))}{(z + \Sigma_{aa}^L(z))(z + \gamma + \Sigma_{bb}(z)) + \omega^2 (1 - \Sigma_{ab}^L(z))^2} \tag{38}$$

Here we have used the following symmetry relation

$$\Sigma_{ba}^L(z) = -\omega^2 \Sigma_{ab}^L(z) \tag{39}$$

which follows from the definition of the kernels Σ_{ab} and Σ_{ba}, (22), and $C_{ab}(t) \equiv K_{i\alpha} < a_i(t) b_\alpha(0) >= K_{i\alpha} < a_i(0) b_\alpha(-t) >= -K_{i\alpha} < b_\alpha(t) a_i(0) >= -C_{ba}(t)$. The first equality is due to the time translation invariance and the second one from the time reversal property of the velocity components. Note that for $\delta^* = 1$ the two correlators $C_a^L(z)$ and $C_a^{KL}(z)$ become identical.

Similarly, from (34)-(35), we obtain

$$C_{ab}^L(z) = \frac{T(1-\Sigma_{ab}^L(z))}{(z+\Sigma_{aa}^L(z))(z+\gamma+\Sigma_{bb}(z))+\omega^2(1-\Sigma_{ab}^L(z))^2} \tag{40}$$

$$C_b^L(z) = \frac{T(z+\Sigma_{aa}^L(z))}{(z+\Sigma_{aa}^L(z))(z+\gamma+\Sigma_{bb}(z))+\omega^2(1-\Sigma_{ab}^L(z))^2} \tag{41}$$

Now let us look at the behavior of the correlators for different values of δ^*. For $\delta^*=0$ the only nonvanishing kernel is $\Sigma_{bb}^L(z)$. Hence we obtain

$$C_a^L(z) = \frac{T}{\omega^2}\frac{1}{z}, \quad \Sigma_{bb}^L(z) = \frac{2g^2\omega^2 T}{z}, \tag{42}$$

$$C_a^{KL}(z) = \frac{T}{\omega^2}\frac{1}{z}\left[1-\frac{\omega^2}{z(z+\gamma)+(1+2g^2T)\omega^2}\right], \tag{43}$$

$$C_b^L(z) = \frac{zT}{z(z+\gamma)+(1+2g^2T)\omega^2}, \tag{44}$$

$$C_{ab}^L(z) = -C_{ba}^L(z) = \frac{T}{z(z+\gamma)+(1+2g^2T)\omega^2} \tag{45}$$

Here we point out that there appears to be a subtlety associated with the two limiting procesures: (A) first take $\delta^* = M/N = 0$ before any calculation. (B) first calculate with $\delta^* > 0$ and then take the limit $\delta^* \to 0+$. The procesure (A) gives both $C_a(t) = C_a(0) = T/\omega^2$ and $C_a^K(t) = C_a^K(0) = T/\omega^2$. This is simply due to the fact that the $\{a\}$ variables are time-independent since there is no velocity variable $\{b\}$ that drives dynamics of $\{a\}$. However the results (42)-(45) were obtained by adopting the second limiting procedure (B). Here $C_a(t)$ is trivially nonergodic: $C_a(t) = C_a(0) = T/\omega^2$ whereas and $C_a^K(t)$ exhibits a nontrivial nonergodic behavior: $C_a^K(t\to\infty) = (T/\omega^2)\cdot 2g^2T/(1+2g^2T)$. The difference between these two procedures can be seen also by looking at (33) for $C_a^{KL}(z)$. The terms except the first one on the right hand side is absent if the first limiting procedure (A) is adopted, whereas it remains finite in the second limiting procedure (B).

For $\delta^* = 1$ where $M = N$ and $K_{i\alpha} = \delta_{i\alpha}$, $C_a^L(z) = C_a^{KL}(z)$ reproduces the equation derived in [14], apart from the wave number dependence. Note that if we put $\Sigma_{aa}^L(z) = \Sigma_{ab}^L(z) = 0$ *by hand*, (36) or (37) gives a closed equation for $C_a(t)$ alone. This equation is nothing but the Leutheusser's schematic MC equation giving a dynamic transition from ergodic phase to nonergodic one. But in reality Σ_{aa} and Σ_{ab} can not be ignored and our numerical solution strongly indicates that the system remains ergodic for all temperatures due to the strong contribution of these so called hopping terms. Furthermore these hopping terms do not become self-consistently small as temperature is lowered. Therefore the density correlator does not show a continuous slowing down with lowering temperature. This result was striking to us since usually a mean-field-type theory, such as the dynamics of the spherical p-spin model in the limit of $N\to\infty$, often gives a sharp dynamic transition. In fact, we were first constructing the toy model with $M = N$ and we expected that the model designed to rigorously reproduce the idealized MCT exhibits such a dynamic transition. But to our surprise the dynamic transition was absent in the N-component toy model. This aspect is a fundamental difference in the two kinds of mean-field-type theories with and without reversible mode coupling. The foremost example of the latter is the sherical p-spin model where the ergodic-to-nonergodic transition is driven by the dissipative nonlinearity which comes from the nonlinear random Hamiltonian. As demonstrated

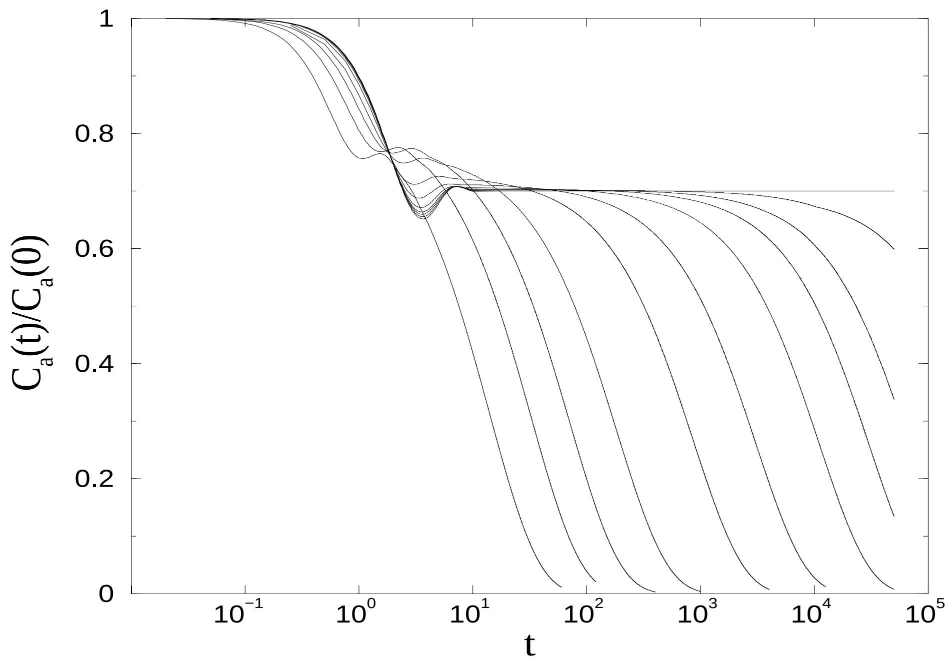

Figure 1. The relaxation of the nonmalized density correlator $C_a(t)/C_a(0)$ for $\delta^* = 0.3$. The other parameters are given by $g = \gamma = \omega = 1$. The curves are, from left to right at long times, for $T = 5, 2, 1, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01$, and 0.001

below, in order to have such a sharp transition in our toy model, we find it necessary to extend the original N-component model to the model with $M < N$. Thus it is very difficult to understand the idealized MCT *without* relying upon uncontrolled approximation. It is also interesting to note that the ergodicity restoring process in our toy model (represented by the kernels Σ_{aa} and Σ_{ab}) has nothing to do with a thermally activated energy barrier crossing since the gaussian Hamiltonian in our model does not possess such a barrier.

Our numerical solution for $\delta^* = 0.3$ is shown in Figure 1 with various values of T. The other parameters were fixed as $\omega = 1$, $\gamma = 1$, and $g = 1$. As T is lowered, the relaxation exhibits a continuous slowing and it appears to be frozen at lowest T. One may ask whether this freezing reflects the presence of the genuine nonergodicity or it is merely apparent: the decaying will be observed if the observation time window is further extended. The question of the existence of nonergodicity is easily answered in the usual idealized MCT where one can easily solve the closed equation for the nonergodicity parameter to obtain the phase diagram. The situation is very different in our toy model. When we expand the correlators as $C_a^L(z) = f_a/z + f_a^{(0)} + f^{(1)}z + \cdots$ etc., we end up with a herarchically connected set of equations for all the f's, which can not be easily analyzed numerically.

An analytic feature signifying the presence of the genuine nonergodic state can be seen by adiabatically eliminating the velocity components in the limit of large γ and obtaining the Fokker-Planck equation for the distribution function $\tilde{D}(\{a\}, t)$

containing only the $\{a\}$ variables:

$$\frac{\partial \tilde{D}(\{a\},t)}{\partial t} = \frac{\partial}{\partial a_i}\left[Q_{ij}(\{a\})\left(\frac{\partial}{\partial a_j} + \frac{\omega^2}{T}a_j\right)\tilde{D}(\{a\},t)\right] \tag{46}$$

Here the diffusion matrix $Q_{ij}(\{a\})$ is given by

$$Q_{ij}(\{a\}) \equiv \frac{T}{\gamma} M_{i\alpha} M_{j\alpha}$$
$$M_{i\alpha} \equiv K_{i\alpha} + \frac{\omega}{\sqrt{N}} J_{ik\alpha} a_k \tag{47}$$

An important point is that the diffusion matrix Q_{ij} is *singular* for $M < N$, i.e., $\det|\mathbf{Q}| = 0$ [15]. The proof is simple. Define a $N \times N$ matrix $\mathbf{M}$ by $\mathbf{M}_{ij} \equiv M_{i,j=\alpha}$ for $j \le M$, $\mathbf{M}_{ij} \equiv 0$ for $j > M$. Then we obtain in matrix notation $\mathbf{Q} = (T/\gamma)\mathbf{M}\cdot\mathbf{M}^T$ (The superscript T denotes the transposed matrix). Then $\det|Q| = (T/\gamma)^N(\det|\mathbf{M}|)^2 = 0$ since $\det|\mathbf{M}| = 0$ by construction. This implies that the Fokker-Planck equation (46) can have *nonequilibrium* stationary solution other than the equilibrium one, $\tilde{D}_e(\{a\}) = cst.\exp(-\omega^2 a_j^2/2T)$. This nonequilibrium stationary solutions are precisely the kind of nonergodic states found numerically in the present toy model. The general stationary solution [16] is given by

$$\tilde{D}_L(\{a\}) = \mathcal{F}(\xi_j a_j)\,\mathrm{e}^{-\frac{\omega^2}{2T}a_i^2} \tag{48}$$

where ξ_i is the eigenvector of the diffusion matrix Q_{ij} with zero eigenvalue. If the function $\mathcal{F}(x)$ is a constant, then $\tilde{D}_L(\{a\}) = \tilde{D}_e(\{a\})$ is the equilibrium distribution, otherwise it is a nonequilibrium stationary distribution.

One instructive case for the nonequilibrium stationary solutions is that of $g = 0$. For this case, Q_{ij} becomes proportional to the dynamic matrix Ω_{ij}: $Q_{ij} = (T/\gamma)K_{i\alpha}K_{j\alpha} = (T/\gamma\omega^2)\Omega_{ij}$. By the same argument as above Ω_{ij} is singular as well. Note from (36) that $C_a^L(z) = (T/\omega^2)\cdot(1-\delta^*)/z$ in the limit of $z \to 0$. The other correlators do not diverge at $z = 0$. Hence the model is nonergodic for $0 \le \delta^* < 1$: the system is always driven into the nonergodic state in the linear case ($g = 0$). In this case the thermal noise alone is not enough to drive the system to the equilibrium state. This case is somewhat reminiscent of the ideal gas case or the collection of independent harmonic oscillators where the systems are trivially nonergodic due to the absence of interactions. Only when the nonlinear reversible mode coupling is present, as T increases, the thermal noise can drive the system to the equilbrium state, hence making the system ergodic. The onset temperature at which the ergodicity is recovered is the dynamic transition temperature.

In any event, further numerical and theoretical studies of possibile ergodic-to-nonergodic transitions for nontrivial case $g \neq 0$ are warranted.

4. Summary

We have constructed a dynamic mean-field-type model involving N-component density and M-component velocity variables with reversible mode coupling and trivial Hamiltonian. The model is exactly solvable in the limit of $N, M \to \infty$ with keeping the ratio $\delta^* \equiv M/N$ finite. The model exhibits a sharp dynamic transition to a nonergodic state only in the range $0 \le \delta^* < 1$. The nature of the nonergodic state can be understood in terms of the nonequilibrium stationary solution of the Fokker-Planck equation for the probability distribution for the density variable. It would be interesting to investigate the nonequilibrium aging behavior of the model.

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More recent developments can be found in the collection of papers in *Transport Theor. Stat. Phys.* 1995 **24** Nos 6-8 and the conference proceedings such as *J. Non-cryst. Solids* 1998 **235-237** and *J. Phys.: Condens. Matter* 1999 **10A**.

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